MARKOVIAN QUEUES WITH ARRIVAL DEPENDENCE

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THESIS

MARKOVIAN QUEUES WITH
ARRIVAL DEPENDENCE

by

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Markovian Queues With
Arrival Dependence

by

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Captain, United States Army
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Submitted in partial fulfillment of the
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A study of three Markovian queues wherein customers require two separate types of service upon arrival. The two service channels operate independently but receive demands through a common arrival process. Transient and steady state results are established in the case that the service channels have an infinite number of servers. The remaining two systems, finite server and finite capacity, are not completely modeled. However, special results concerning their stochastic nature are documented.

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I. INTRODUCTION

The theory of queues is rich with information on the classical type of queueing problems. In most cases, the solution to these problems is straightforward and agrees with one's intuitions concerning actual physical situations. Quite often, however, solutions rely upon the fact that arrival and service distributions are independent. In n-server queues, one usually stipulates that arrivals enter each queue independently.

The purpose of this thesis is to investigate three types of service systems in which arriving customers generate jobs to two service systems simultaneously. Thus, while the service systems work independently of each other, they both receive customers via the same arrival mechanism.

The first system to be studied is the two channel queue, where each service channel has an infinite number of servers. The second system is a modification of the first whereby each service channel has only one server and queues are allowed to build up. Finally, the third system further modifies the situation by assuming that the entire system has a maximum capacity of K customers.

The following chapters will show that the infinite server system is easily solved. However, as soon as one restricts the number of servers to one per channel, several difficulties arise. At this time, only partial information has been

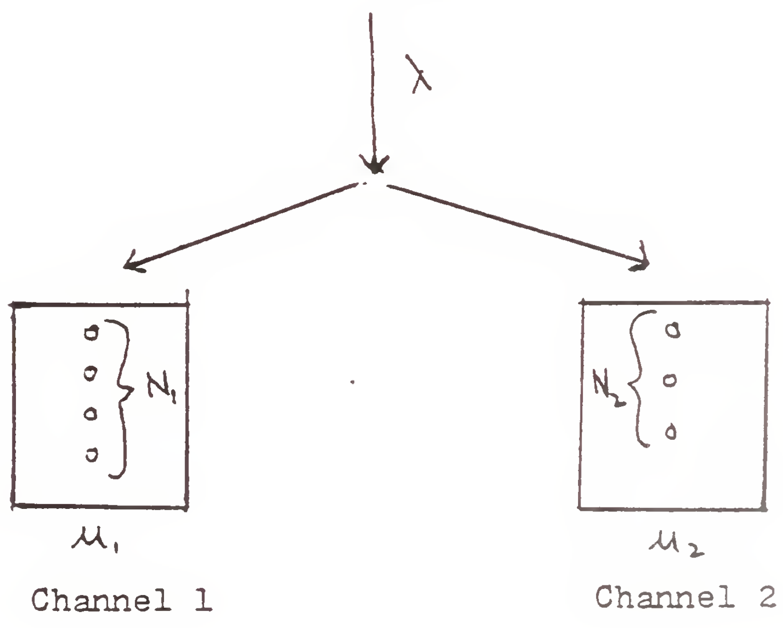
established concerning the probabalistic nature of these two later systems. Hopefully, further investigation can be performed utilizing the current findings, limited as they are.

II. THE TWO CHANNEL INFINITE SERVER QUEUE

In this first case, it is assumed that customer arrivals occur in accordance with a Poisson Process of intensity λ. Each arrival requires two separate services which are performed simultaneously. The two service channels operate independently and each have an infinite or unlimited number of servers. The servers in channel one work at an exponential rate with mean $\frac{1}{\mu_1}$, while those in channel two work at an exponential rate with mean $\frac{1}{\mu_2}$.

Let N be the number of customers in the system at equilibrium. Let N_1 and N_2 be the number of jobs being performed in each channel at equilibrium.

The diagram below shows the system for an arbitrary instant in time.



Note that $N = \max \{N_1, N_2\}$ and that N_1 and N_2 are dependent random variables. Furthermore, because the service channels are independent the marginal distributions for N_1 and N_2 are both Poisson.

$$P(N_1 = i) = \frac{e^{-\frac{\lambda}{\mu_1}} \left(\frac{\lambda}{\mu_1}\right)^i}{i!}$$

$$P(N_2 = j) = \frac{e^{-\frac{\lambda}{\mu_2}} \left(\frac{\lambda}{\mu_2}\right)^j}{j!}$$

In order to describe the characteristics of this queueing system, it will be necessary to apply the theory of the filtered Poisson Process. Parzen defines a filtered Poisson Process as a stochastic process which can be represented as

$$X(t) = \sum_{n=0}^{A(t)} w(t, \tau_n, Y_n)$$

where $A(t)$ is a Poisson Process with intensity λ.

Y_n is a sequence of independent random variables identically distributed as Y and independent of $A(t)$.

$w(t, \tau_n, Y_n)$ is a response function and τ_n is the time of the n^{th} arrival. Here w indicates whether an arrival is still present in the system at time t.

In the context of queueing theory, Y_n will be the sequence of service time random variables. The response function will be defined as

$$w(t, \tau_n, Y_n) = \begin{matrix} 1 & \text{iff } Y_n & t-t_n \\ 0 & \text{iff } Y_n & t-t_n \end{matrix}$$

In effect, w "counts" the number of customers present at time t. Thus the number of customers present can be viewed as

$$N(t) = \sum_{n=0}^{A(t)} w(t, \tau_n, Y_n)$$

A. STEADY STATE DISTRIBUTION FOR THE NUMBER OF CUSTOMERS IN THE SYSTEM

Since both services commence simulaneously, the service time for each customer is $Y_n = \max\{S_1, S_2\}$ where $S_1 \sim$ Exponential (μ_1) and $S_2 \sim$ Exponential (μ_2), S_1 and S_2 independent. Thus for all n,

$$P(Y_n < t) = P(S_1 < t) \quad P(S_2 < t)$$

$$P(Y_n < t) = (1 - e^{-\mu_1 t})(1 - e^{-\mu_2 t})$$

Parzen shows that under the above conditions $N(t)$ is Poisson distributed with mean $\lambda \int_0^t (1-F_y(s))\, ds$. Thus, $N(t)$ has mean $\lambda \int_0^t (e^{-\theta_1 s} + e^{-\theta_2 s} - e^{-(\theta_1 + \theta_2)s})ds$. In the limit, N is Poisson distributed with mean $E(Y)$ where $E(Y) = \frac{1}{\mu_1} + \frac{1}{\mu_2}$ $- \frac{1}{\mu_1 + \mu_2}$

After some algebraic simplification, the mean is

$E(N) = \dfrac{\lambda(\mu_2^2 + \mu_1^2 + \mu_1 \mu_2)}{\mu_1 \mu_2 (\mu_1 + \mu_2)}$. One notes that $E(N)$ is

less than the sum of $E(N_1)$ and $E(N_2)$, which is to be expected.

B. JOINT MOMENT GENERATING FUNCTION FOR N_1 AND N_2

Given n arrivals in (0,t), the arrival times are distributed as n order statistics from a uniform (0,t) random variable.

Since the order of arrival has no impact on service (infinite server queue), the probability that an arriving customer will still be present at time t is merely a function of the service distribution and the time of arrival.

Consider an arbitrary arrival, j, in (0,t). The arrival generates one job for each service channel. Let $Y_{1j}(t)$, $Y_{2j}(t)$ be a two dimensional, bivariate Bernoulli random variable where the event $Y_{1j}(t) = 1$ is defined as the j job in channel 1 being present at time t. $Y_{2j}(t) = 1$ is defined as the event where the j job is present in channel 2.

Now for the arbitrary arrival, let x be arrival time. Thus $x \sim$ Uniform (0,t) and the joint moment generating function for $Y_{1j}(t)$, $Y_{2j}(t)$ is

$$g_{Y_{1j}(t),\, Y_{2j}(t)}(\theta_1, \theta_2) = (p_1 + q_1 e^{\theta_1})(p_2 + q_2 e^{\theta_2})$$

where $p_1 = 1 - F_1(t-x)$, $p_1 + q_1 = 1$

$p_2 = 1 - F_2(t-x)$, $p_2 + q_2 = 1$

and F_1 and F_2 are the service time distribution functions for channels 1 and 2 respectively.

Thus, unconditioning on the arrival time X,

$$g_{Y_{1j}(t),Y_{2j}(t)}(\theta_1,\theta_2) = \frac{1}{t}\int_0^t (1-F_1(t-x) + F_1(t-x)e^{\theta_1})\cdot$$

$$(1 - F_2(t-x) + F_2(t-y)e^{\theta_2})\cdot dx$$

Letting w=t-x for a change of integration variable

$$g_{Y_{1j}(t),Y_{2j}(t)}(\theta_1,\theta_2) = \frac{1}{t}\int_0^t (1-F_1(w) + F_1(w)e^{\theta_1})\cdot$$

$$(1-F_2(w) + F_2(w)e^{\theta_2})\ dw$$

Considering all n arrivals in (0,t), each arrival creates an independent bivariate Bernoulli random variable. Hence, letting

$$X_1(t) = \sum_{j=1}^{n} Y_{1j}(t) \qquad X_2(t) = \sum_{j=1}^{n} Y_{2j}(t)$$

then the joint moment generating function for $X_1(t)$ and $X_2(t)$, conditional on n arrivals in (0,t), is

$$f_{X_1(t),\ X_2(t)}(\theta_1,\theta_2) = \left[g_{Y_{1j}(t),\ Y_{2j}(t)}(\theta_1,\theta_2)\right]^n$$

$$f_{X_1(t),\ X_2(t)}(\theta_1,\theta_2) = \frac{1}{t^n}\left[\int_0^t (1-F_1(w) + F_1(w)e^{\theta_1})\cdot\right.$$

$$\left.(1-F_2(w) + F_2(w)e^{\theta_2})dw\right]^n$$

Thus letting $N_1(t)$, $N_2(t)$ be the numbers present at time t for any number of arrivals in $(0,t)$

$$M_{N_1(t),\, N_2(t)}(\theta_1,\theta_2) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} f_{X_1(t),\, X_2(t)}(\theta_1,\theta_2)$$

It follows that,

$$M_{N_1(t),\, N_2(t)}(\theta_1,\theta_2) = e^{-\lambda t} \quad e^{A}$$

$$\text{where } A = \int_0^t (1-F_1(w)+F_1(w)e^{\theta_1})(1-F_2(w)+F_2(w)e^{\theta_2})\,dw$$

Letting Y_1 and Y_2 be Bernoulli random variables, specific-cally

$Y_1 \sim$ Bernoulli (p_1) where $p_1 = 1-F_1(w)$

$Y_2 \sim$ Bernoulli (p_2) where $p_2 = 1-F_2(w)$,

then the moment generating function can be expressed as

$$M_{N_1(t),\, N_2(t)}(\theta_1,\theta_2) = e^{\lambda\left[a(e^{\theta_1+\theta_2}-1) + b(e^{\theta_1}-1) + c(e^{\theta_2}-1)\right]}$$

where

$$a = \int_0^t p_1 p_2 \, dw \qquad b = \int_0^t p_1 q_2 \, dw \qquad c = \int_0^t p_2 q_1 \, dw$$

Now consider the limiting form of the above function as t .

First

$$\lim_{t \to \infty} a = e^{-\mu_1 w} e^{-\mu_2 w} \, dw = \frac{1}{\mu_1 + \mu_2}$$

$$\lim_{t \to \infty} b = e^{-\mu_1 w}(1-e^{-\mu_2 w}) \, dw = \frac{1}{\mu_1} - \frac{1}{\mu_1 + \mu_2} = \frac{\mu_2}{\mu_1(\mu_1 + \mu_2)}$$

$$\lim_{t \to \infty} c = e^{-\mu_2 w}(1-e^{-\mu_1 w}) \, dw = \frac{1}{\mu_2} - \frac{1}{\mu_1 + \mu_2} = \frac{\mu_1}{\mu_2(\mu_1 + \mu_2)}$$

Then

$$M_{N_1, N_2}(\theta_1, \theta_2) = \lim_{t \to \infty} M_{N_1(t), N_2(t)}(\theta_1, \theta_2) =$$

$$e^{\lambda \left[\frac{1}{\mu_1 + \mu_2} (e^{\theta_1 + \theta_2} - 1) + \frac{\mu_2}{\mu_1(\mu_1 + \mu_2)} (e^{\theta_1} - 1) + \frac{\mu_1}{\mu_2(\mu_1 + \mu_2)} (e^{\theta_2} - 1) \right]}$$

It should be noted that the marginal generating functions, and hence, N_1, N_2 are Poisson.

$$M_{N_1}(\theta_1) = M_{N_1,N_2}(\theta_1,0) = e^{\frac{\lambda}{\mu_1}(e^{\theta_1} - 1)}$$

$$M_{N_2}(\theta_2) = M_{N_1,N_2}(0,\theta_2) = e^{\frac{\lambda}{\mu_2}(e^{\theta_2} - 1)}$$

Now consider the second mixed moment, $E\left[N_1N_2\right]$.

$$E\left[N_1N_2\right] = \lim_{\substack{\theta_1 \to 0 \\ \theta_2 \to 0}} \frac{\partial^2 M}{\partial\theta_1 \partial\theta_2}$$

$$\frac{\partial M}{\partial \theta_1} = \lambda\left(\frac{1}{\mu_1 + \mu_2} e^{\theta_1+\theta_2} + \frac{\mu_2}{\mu_1(\mu_1+\mu_2)} e^{\theta_1}\right) M_{N_1,N_2}(\theta_1,\theta_2)$$

and $$\frac{\partial^2 M}{\partial\theta_1 \partial \theta_2} = \lambda\left(\frac{1}{\mu_1+\mu_2} e^{\theta_1+\theta_2} + \frac{\mu_2}{\mu_1(\mu_1+\mu_2)} e^{\theta_1}\right) \frac{\partial M}{\partial \theta_2}$$

$$+ \frac{\lambda}{\mu_1 + \mu_2} e^{\theta_1+\theta_2} M_{N_1,N_2}(\theta_1,\theta_2)$$

Therefore

$$E\left[N_1N_2\right] = \frac{\lambda}{\mu_1} \lim_{\theta_2 \to 0} \frac{\partial M}{\partial \theta_2} + \frac{\lambda}{\mu_1 + \mu_2}$$

Since $N_1 \sim$ Poisson $\frac{\lambda}{\mu_1}$ and $N_2 \sim$ Poisson $\frac{\lambda}{\mu_2}$

then $\lim_{\theta_2 \to 0} \frac{\partial M}{\partial \theta_2} = E[N_2] = \frac{\lambda}{\mu_2}$

and $E[N_1] = \frac{\lambda}{\mu_1}$

Thus, the correlation between N_1 and N_2 is

$$\rho = \frac{E(N_1N_2) - E(N_1)\,E(N_2)}{\sqrt{V(N_1)\ V(N_2)}}$$

$$\rho = \frac{\frac{\lambda^2}{\mu_1\mu_2} + \frac{\lambda}{\mu_1+\mu_2} - \frac{\lambda}{\mu_1}\left(\frac{\lambda}{\mu_2}\right)}{\sqrt{\frac{\lambda}{\mu_1}\quad\frac{\lambda}{\mu_2}}}$$

$$\rho = \frac{\sqrt{\mu_1\ \mu_2}}{\mu_1+\mu_2}\,,$$ which is independent of λ.

III. THE TWO CHANNEL QUEUE WITH A SINGLE SERVER IN EACH QUEUE

As a modification to the first system, each service channel will now be limited to a single server. The servers in channels one and two work with exponential rates μ_1 and μ_2, respectively. As before, N is the number of customers in the system at steady state. N_1 and N_2 are the number of jobs to be done in each channel. The jobs are served using a first in-first out discipline. The arrivals are generated by a Poisson Process of intenstiy λ.

A. STEADY STATE BALANCE EQUATIONS FOR THE SYSTEM

Since the system has both exponential interarrival and exponential service times, the balance equations for the two-dimensional steady state probabilities can be expressed as follows. Letting $P_{n_1,n_2} = P(N_1=n_1, N_2=n_2)$ one has that

$$(\lambda+\mu_1+\mu_2)P_{n_1n_2} = \lambda P_{n_1-1,\ n_2-1} + \mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1}$$

$$1 \leq n_1 < \infty$$

$$1 \leq n_2 < \infty$$

$$(\lambda+\mu_1) P_{n_1,0} = \mu_1 P_{n_1+1,0} + \mu_2 P_{n_1,1}$$

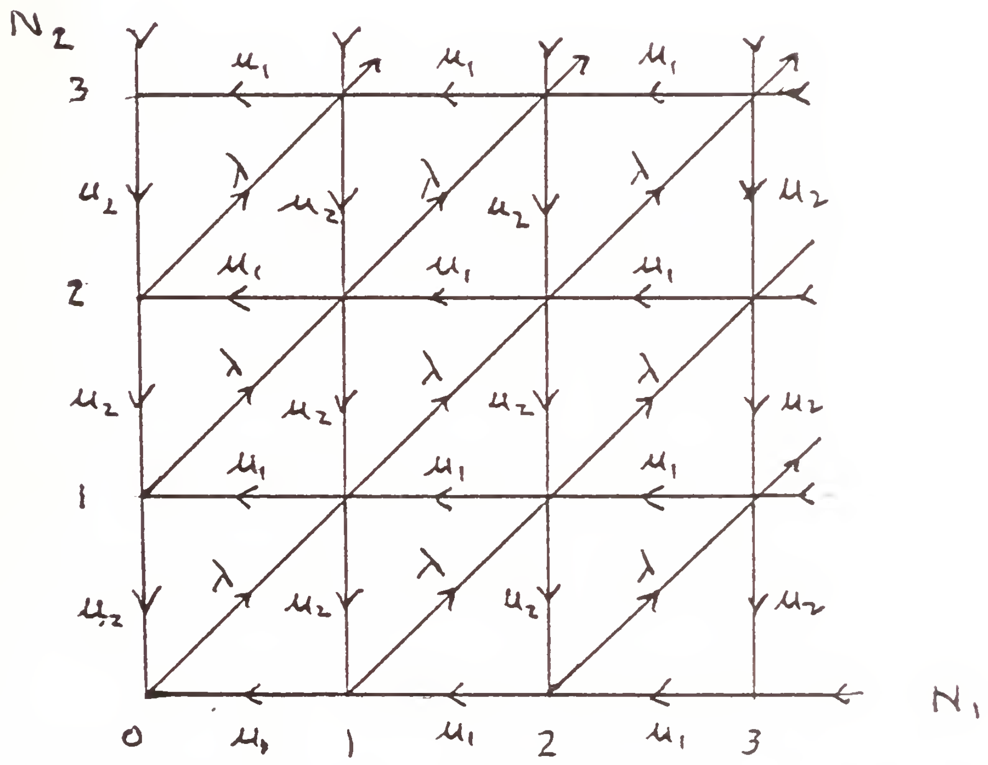
$$1 \leq n_1 < \infty$$

$$(\lambda+\mu_2) P_{0,n_2} = \mu_2 P_{0,n_2+1} + \mu_1 P_{1,n_2}$$

$1 \leq n_2 < \infty$

$$\lambda P_{00} = \mu_1 P_{10} + \mu_2 P_{01}$$

The transition's between states can best be visualized using the following transition diagram



One might suppose that the balance equation could be solved using difference equation techniques. This does not seem to be the case. First, most techniques used to solve partial difference equations of the above type assume a solution of the form,

$$P_{n_1,n_2} = a^{n_1} b^{n_2}$$

Under the above assumption, one could solve for P_{n_1,n_2} and express the result as

$$P_{n_1,n_2} = \sum_{j=1}^{K} c_j \, a_j^{\,n_1} \, b_j^{\,n_2} \quad \text{where } (a_j, b_j)$$

$$j = 1, 2, \ldots K$$

are the K roots of the characteristic equation and c_j's are arbitrary constants. However, since P_{n_1,n_2} is a function of two dependent random variables N_1 and N_2, it is known that P_{n_1,n_2} cannot be factored into a separable product similar to $a^{n_1} b^{n_2}$. If this were the case, then N_1 and N_2 would be independent.

Therefore, for each probability there are two or more arbitrary constants to be evaluated. If one solves the major difference equation for the interior points of the first quadrant, i.e. assume $P_{n_1,n_2} = a^{n_1} b^{n_2}$

$$(\lambda + \mu_1 + \mu_2) \, P_{n_1 n_2} = \lambda P_{n_1-1,n_2-1} + \mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1}$$

the roots are $a=1 \quad b=1, \quad a=1 \quad f=\dfrac{\lambda}{\mu_2}$

and $a = \dfrac{\lambda}{\mu_1} \quad b=1$. Thus the general solution is

$$P_{n_1,n_2} = \sum_{j=1}^{3} c_j(n_1,n_2) \, a_j^{\,n_1} \, b_j^{\,n_2}$$

or

$$P_{n_1,n_2} = c_1(n_1,n_2) + c_2(n_1,n_2)\left(\frac{\lambda}{\mu_2}\right)^{n_2} + c_3(n_1,n_2)\left(\frac{\lambda}{\mu_1}\right)^{n_1}$$

Even using all of the boundary conditions, one cannot find a particular solution to the above equation.

One point to note is that $P_{n_1 n_2}$ can be expressed as

$$P_{n_1 n_2} = P_{n_1} + P_{n_2} - P(n_1 \cup n_2)$$

Thus if one assumes that $C_2(n_1,n_2)$ and $C_3(n_1,n_2)$ are actually $(1 - \frac{\lambda}{\mu_2})$ and $(1 - \frac{\lambda}{\mu_1})$ respectively, the particular solution takes the following form

$$P_{n_1,n_2} = c_1(n_1,n_2) + \left(1 - \frac{\lambda}{\mu_2}\right)\left(\frac{\lambda}{\mu_2}\right)^{n_2} + \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^{n_1}$$

Again, using the boundary conditions only results in a system of equations for $C(n_1,n_2)$ which is equivalent to the original set for P_{n_1,n_2}.

It appears as though the boundary conditions are of a non-separable type, and consequently there is no explicit solution for the balance equations.

B. SOLUTION TO THE BALANCE EQUATIONS USING THE PROBABILITY GENERATING FUNCTION

The joint probability generating function can be expressed as

$$G(s,t) = \sum_{n_1=0} \sum_{n_2=0} s^{n_1} t^{n_2} P_{n_1,n_2}$$

Using the balance equations one can derive the following expression for the joint generating function.

$$G(s,t) = \frac{t(s-1)\sum\limits_{n_2=0} P_{0,n_2} + s(t-1)\sum\limits_{n_1=0} P_{n_1,0}}{(\lambda+\mu_1+\mu_2)\,st - \lambda s^2t^2 - t\mu_1 - s\mu_2}$$

In other words, G(s,t) is expressed as a function of s, t, and the boundary probabilities. If one knew these probabilities, G(s,t) could be solved in closed form. However such is not the case. It should be pointed out that the same type of equational form occurs when finding a generating function for queues which have either Gamma interarrival or Gamma service times. However, the boundary probabilities are finite in number and can be solved for using Rouche's theorems on the roots of complex functions. It appears as though this method is fruitless for the above function.

As in the infinite server queue of Chapter 2, the marginal distributions, $f_{N_1}(n_1)$ and $f_{N_2}(n_2)$ are known. They are geometric and can be expressed as

$$f_{N_1}(n_1) = \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^{n_1} \qquad n_1 = 0,\ 1,\ 2,\ \ldots$$

$$f_{N_2}(n_2) = \left(1 - \frac{\lambda}{\mu_2}\right)\left(\frac{\lambda}{\mu_2}\right)^{n_2} \qquad n_2 = 0,\ 1,\ 2,\ \ldots$$

As one would expect, the marginal generating functions are also geometric.

$$G(1,t) = \frac{(t-1)\mu_2 \sum_{n_1=0} P_{n_1,0}}{(\lambda+\mu_1+\mu_2)\,t - \lambda t^2 - t\mu_1 - \mu_2}$$

$$G(1,t) = \frac{(t-1)\mu_2 \left(1-\frac{\lambda}{\mu_2}\right)}{-\lambda t^2+(\lambda+\mu_2)t-\mu_2} = \frac{(t-1)\mu_2 \left(1-\frac{\lambda}{\mu_2}\right)}{-\lambda t(t-1)+\mu_2(t-1)}$$

$$G(1,t) = \frac{\mu_2\left(1-\frac{\lambda}{\mu_2}\right)}{-\lambda t+\mu_2} = \frac{1-\frac{\lambda}{\mu_2}}{1-\frac{\lambda}{\mu_2}\,t}$$

which is the generating function for a geometric random variable with parameter $1 - \frac{\lambda}{\mu_2}$

Likewise $G(s,1) = \dfrac{1-\frac{\lambda}{\mu_1}}{1-\frac{\lambda}{\mu_1}\,s}$

Thus, little has been gained using the generating function approach.

C. AN ATTEMPT TO FIND THE JOINT GENERATING FUNCTION BY CONDITIONING ON ARRIVING CUSTOMERS

In Chapter 2, the joint generating function of N_1 and N_2 was found using a filtered Poisson Process. In essence, the method worked because the stochastic dependence of $N_1(t)$ and $N_2(t)$ was handled by conditioning on both the number and arrival times of n customers, given an arbitrary time t since the process started. Thus $N_1(t)$ and $N_2(t)$, conditioned on

the above events, only depended upon their respective service distributions.

A possible approach to finding the joint moment generating function for $N_1(t)$ and $N_2(t)$ is to handle the stochastic dependence by the same conditioning. Thus, given the number of arrivals at time t,

$$E\left[e^{\mu_1 N_1(t) + \mu_2 N_2(t)}\right] = \sum_{n=0} E\left[e^{\mu_1 N_1(t) + \mu_2 N_2(t)} / A(t) = n\right] \cdot P\left[A(t) = n\right]$$

Letting $\emptyset = E\left[e^{\mu_1 N_1(t) + \mu_2 N_2(t)} / A(t) = n\right]$

it follows that

$$\emptyset = \int_0^t \int_{S_1}^t \cdots \int_{S_{n-1}}^t E\left[e^{\mu_1 N_1(t) + \mu_2 N_2(t)} / A(t) = n, S_1, S_2, \ldots S_n\right] \cdot \frac{n!}{t^n} ds_1\, ds_2 \ldots ds_n$$

$$\emptyset = \int_0^t \int_{S_1}^t \cdots \int_{S_{n-1}}^t E\left[e^{\mu_1 N_1(t)} / N(t) = n_1 S_1 S_1 \ldots S_n\right] \cdot E\left[e^{\mu_2 N_2(t)} / N(t) = n_1 S_1 \ldots S_n\right] \times \frac{n!}{t^n} ds_1 \ldots ds_n$$

Now the problem is to find the moment generating function for $N_1(t)$ and $N_2(t)$ conditioned on the two given events. It appears that this is not a trivial problem.

One might recall that in the infinite server queues, $N_1(t)$ and $N_2(t)$, conditioned on the same events, were actually binomially distributed with

$$P = \int_0^t \frac{e^{-\mu_1(t-\tau)}}{t} \, d\tau$$

The probability that any customer would be present at t was independent of the presence of other customers. However, with a waiting line in each queue, a dependence is generated. If the j^{th} arrival is present at time t, so are the arrivals $j+1, j+2, \ldots n$.

To this date, the conditional distribution of $N_1(t)$ (or $N_2(t)$) has not been determined. One point to consider is that whatever the conditional distribution is, it must be such that the limiting distribution of $N_1(t)$ (or $N_2(t)$) is geometric.

It should also be observed that the established theorems concerning a filtered Poisson Process do not apply when waiting lines develop. Even though $N_1(t)$ can be expressed as

$$N_1(t) = \sum_{j=1}^{A(t)} w_j(t_1, \tau_1, s_1)$$

the response functions are correlated. If $w_j = 1$ for the j^{th} customer then

$w_i = 1$ for $j < i \leq n$.

D. ANALYSIS OF ABSORPTION PROBABILITIES USING RANDOM WALK THEORY

Consider the situation where the state of the system is given. That is, at the present time, $N_1 = i \quad N_2 = j$. What is the probability that the system will reach one section of the boundary before the other section?

The attempt to answer this question used the theory of a random walk. The problem reduces to solving an infinite system of difference equations of the form

$$P_{n_1,n_2} = \frac{\lambda}{\lambda + \mu_1 + \mu_2} P_{n_1-1,\, n_2-1} + \frac{\mu_1}{\lambda + \mu_1 + \mu_2} P_{n_1+1,\, n_2} + \frac{\mu_2}{\lambda + \mu_1 + \mu_2} P_{n_1,n_2+1}$$

for $1 \leq n_1 < \infty \qquad 1 \leq n_2 < \infty$

In this case, the boundary is partitioned into two classes, following an approach introduced by Feller in <u>An Introduction to Probability Theory and Its Applications, Volume I</u>. Using this method of analysis, the boundary points are either given values of 1 or 0. Thus if $P_{n_1,0}$ of $P_{0,n_2} = 1$, the probability of being absorbed by this part of the boundary is 1. Otherwise, the absorption will take place at the other boundary points.

Unfortunately, the solution to this system is as obscure as the original system of balance equations explored earlier. At this point, no solution has been found.

E. SOME ADDITIONAL FACTS CONCERNING THE TRANSIENT DISTRIBUTION OF WAITING TIMES FOR ARRIVING CUSTOMERS

The following material uses the queueing theory approach espoused by Feller in An Introduction to Probability Theory and Its Applications, Volume II. The intent is to develop a closed form expression for the waiting time of the j^{th} arrival, given that n customers had arrived at time t.

It will be noted that the theoretical results are highly intractable. Perhaps further research will simplify the following relationships.

Let,

$$X_n = B_{n-1} - A_n$$

$$S_n = X_1 + X_2 + X_3 + \dots X_n$$

$$S_n = \sum_{i=0}^{n-1} B_i - \sum_{j=1}^{n} A_j \quad = B^* - A^*$$

n = number of customers - (start with customer 0)

A_n = time between arrival of $n-1^{st}$ and n^{th} customer

B_n = service time of n^{th} customer.

W_n = waiting time of n^{th} customer

$$W_n = \max \{0, S_1, S_2, \dots S_n\}$$

If S_k is max $\{0, S_1, S_2, \dots S_n\}$

then $S_k > 0,\ S_k > S_1,\ S_k > S_2, \dots S_k > S_{k-1},$

$$S_{k+1} < S_k,\ S_{k+2} < S_k, \dots S_{n-1} < S_k,\ S_n < S_k$$

The event $\{S_k > 0, S_k > S_1, \ldots S_k > S_{k-1}\}$ is independent of $\{S_{k+1} < S_k, S_{k+2} < S_k, \ldots S_{n-1} < S_k, S_n < S_k\}$

Also the event

$\{S_k > 0, S_k > S_1, \ldots S_k > S_{k-1}\}$ is equivalent to $\{S_1 > 0, S_2 > 0, \ldots S_k > 0\}$

the event

$$\{S_{k+1} - S_k < 0, S_{k+2} - S_k < 0, \ldots S_{n-1} - S_k < 0, S_n - S_k < 0\}$$

is equivalent to

$$\{S_1 < 0, S_2 < 0, S_3 < 0, \ldots S_{n-k-1} < 0, S_{n-k} < 0\}$$

Thus, S_k is max $\{0, S_1, S_2, \ldots S_n\}$

if the two independent events

$$\{S_1 > 0, S_2 > 0, \ldots S_k > 0\} \; \{S_1 < 0, S_2 < 0, \ldots S_{n-k} < 0\}$$

are true.

Let $p_k = P\{S_1 > 0, S_2 > 0, \ldots S_k > 0\}$ $K=0, 1, \ldots n$

$q_{n-k} = P\{S_1 < 0, S_2 < 0, \ldots S_{n-k} < 0\}$ $p_0 = 1$ $q_0 = 1$

Then $P(S_k \text{ is max}) = p_k \, q_{n-k}$

Thus the distribution of $W_n = \max\{0, S_1, S_2 \ldots S_n\}$

is $P(W_n < t) = \sum_{i=1}^{n} P(S_i < t) \; P(S_i \text{ is max})$

or $W_n = 0$ if 0 is max.

Therefore

$$P(W_n < t) = \sum_{i=1}^{n} P(S_i < t)\; p_i\; q_{n-i}$$

$$P(W_n = 0) = q_n$$

Consider $S_k = \sum_{i=1}^{k} (B_{i-1} - A_i) = B^*_{k-1} - A^*_k$

$$P(S_k > 0) = P(B^*_{k-1} > A^*_k)$$

$B^*_{k-1} \sim \text{Gamma}\ (K_1, \mu_1)$

$A^*_k \sim \text{Gamma}\ (K_1, \lambda)$

$$P(S_k > 0) = \int_0^\infty P(B_{k-1} > t)\; f_{A^*_K}(t)\; dt$$

$$P(S_k > 0) = \int_0^\infty \sum_{j=0}^{k-1} e^{-\mu_1 t} \frac{(\mu_1 t)}{j!} \frac{\lambda}{(k-1)!} \cdot (\lambda t)^{k-1} e^{-\lambda t}\, dt$$

$$P(S_k > 0) = \sum_{j=0}^{k-1} \frac{\mu_1}{j!} \frac{\lambda^k}{(k-1)!} \frac{(k+j-1)!}{(\lambda+\mu_1)^{k+j}} \cdot$$

$$\int_0^\infty \frac{\lambda+\mu_1}{(k+j-1)!} \left[(\lambda+\mu_1)t\right]^{k-j+1} e^{-(\lambda+\mu_1)t}\, dt$$

$$P(S_k > 0) = \sum_{j=0}^{k-1} \binom{k+j-1}{k-1} \left(\frac{\mu_1}{\lambda+\mu_1}\right)^j \left(\frac{\lambda}{\lambda+\mu_1}\right)^k$$

Recognizing this to be the CDF of a type of Negative Binomial distribution,

$P(S_k > 0) = P(Z_k < K)$ where $Z \sim$ Negative Binomial with

$$\text{parameters } P^* = \left(\frac{\lambda}{\lambda + \mu_1}\right)$$

and K

$$\text{i.e. } f_Z(z) = \binom{K+z-1}{K-1} \left(\frac{\mu_1}{\lambda + \mu_1}\right)^z \left(\frac{\lambda}{\lambda + \mu_1}\right)^K$$

$$z = 0,\ 1,\ 2,\ \ldots\ldots$$

Note also that Z_k can be expressed as the sum of independent geometric random variables.

$$Z_k = X_1 + X_2 + \ldots\ldots X_k$$

$$\text{where } f_{X_i}(x) = \frac{\lambda}{\lambda + \mu_1} \left(\frac{\mu_1}{\lambda + \mu_1}\right)^x$$

$$x = 0,\ 1,\ 2,\ \ldots\ldots$$

By symmetry, the event $P(S_k < 0)$ can be expressed as $P(B^*_{k-1} < A^*_k) = P(A^*_k > B^*_{k-1})$

thus $P(S_k < 0) = P(W_k < K)$

where $W_k \sim$ Negative Binomial with parameters $P^* = \dfrac{\mu_1}{\lambda + \mu_1}$ and K.

W_k can be expressed as the sum of K independent geometric random variables.

$W_k = Y_1 + Y_2 + \ldots Y_k$

$$\text{where } f_{Y_i}(y) = \frac{\mu_1}{\lambda + \mu_1} \left(\frac{\lambda}{\lambda + \mu_1} \right)^y$$

$$y = 0,\ 1,\ 2,\ \ldots .$$

Now given a sequence

$0,\ S_1\ S_2,\ \ldots .\ S_n$

consider computing $P(S_k \text{ is max})$

$$P(S_k \text{ is max}) = p_k q_{n-k}$$

$$p_k = P(S_1 > 0,\ S_2 > 0,\ \ldots . S_k > 0) = P(Z_1 < 1,\ Z_2 < 2,\ \ldots . Z_k < K)$$

$$p_k = P(X_1 < 1,\ X_1 + X_2 < 2,\ X_1 + X_2 + X_3 < 3,\ \ldots .\ X_1 + X_2 + X_3 + \ldots . X_k < K)$$

The last event can be described as a sequential modified random walk where $X_i = 0,\ 1,\ 2,\ \ldots .$ and the event can only occur when each partial sum is less than K.

As an example:

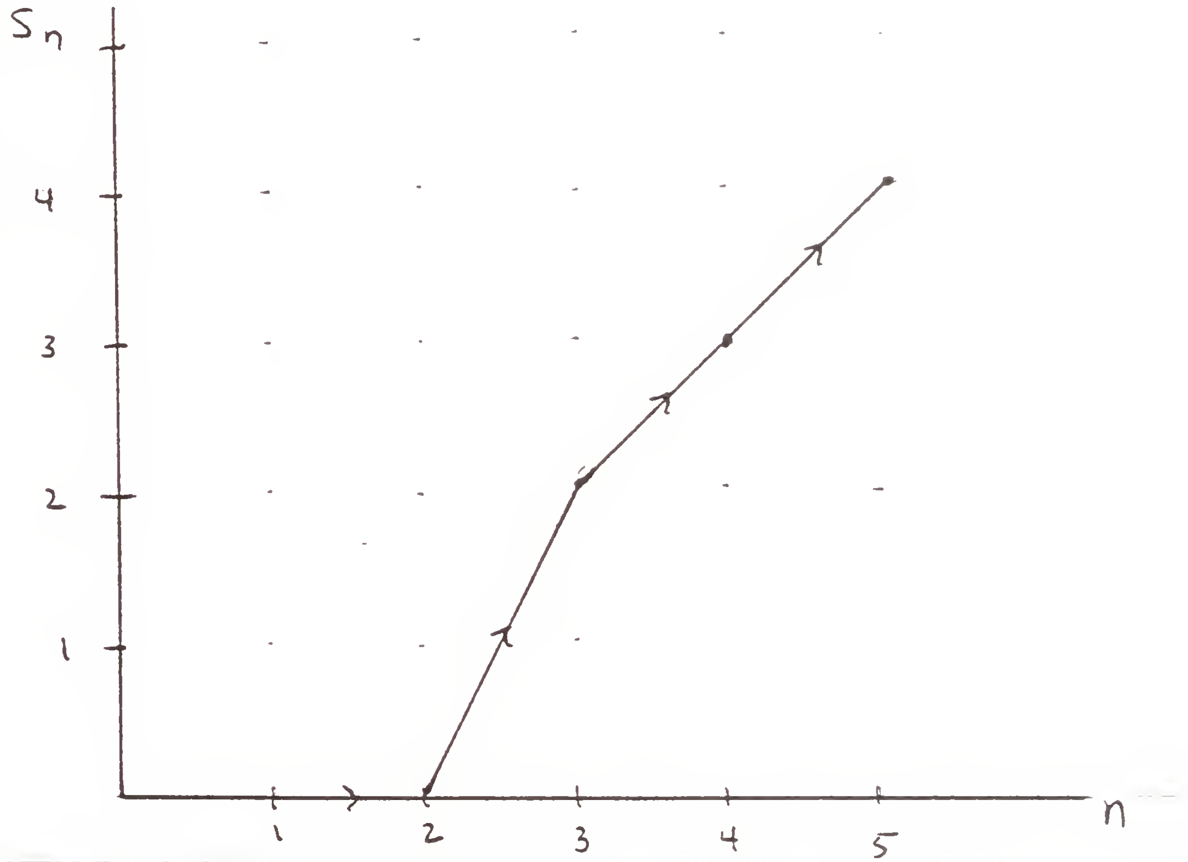
X_1 must $= 0$

X_2 can equal 0 or 1

since $X_1 + X_2 < 2$

etc.

A pictorial representation of the above sequence is shown below for $K = 5$



A sample sequence would be

$X_1 = 0$

$X_2 = 0$

$X_3 = 2$

$X_4 = 1$

$X_5 = 1$.

This sequence satifies the event

$X_1 < 1$, $X_1+X_2 < 2$, $X_1+X_2+X_3 < 3$, $X_1+X_2+X_3+X_4 < 4$

$X_1+X_2+X_3+X_4+X_5 < 5$

In order to calculate the probability of the event $K=5$, i,e.

$$P_5 = P(X_1 < 1,\ X_1+X_2 < 2,\ \ldots.\ X_1+X_2+X_3+X_4+X_5 < 5)$$

one must calculate the probability for each path starting at (1,0) and terminating at either (5,0) (5,1) (5,2) (5,3) while remaining in the acceptable region of transitions.

It so happens that there is a pattern to these paths.

In order to terminate at (5,0)

all $X_i = 0 \quad P((1,0) \rightarrow (5,0)) = \alpha^4$

$$\text{where } \alpha = \frac{\lambda}{\lambda + \mu_1}$$

In order to terminate at (5,1) all $X_i = 0$ except for one:

$P((1,0) \rightarrow (5,1)) = \alpha^4(1-\alpha)$ One must multiply this probability by a factor reflecting the number of paths from $(1,0) \rightarrow (5,1)$. This will be considered shortly.

The other termination probabilities are

$(1,0) \longrightarrow (5,2)$: $\alpha^4(1-\alpha)^2$

$(1,0) \rightarrow (5,3)$: $\alpha^4(1-\alpha)^3$

$(1,0) \rightarrow (5,4)$: $\alpha^4(1-\alpha)^4$

Thus we have shown that

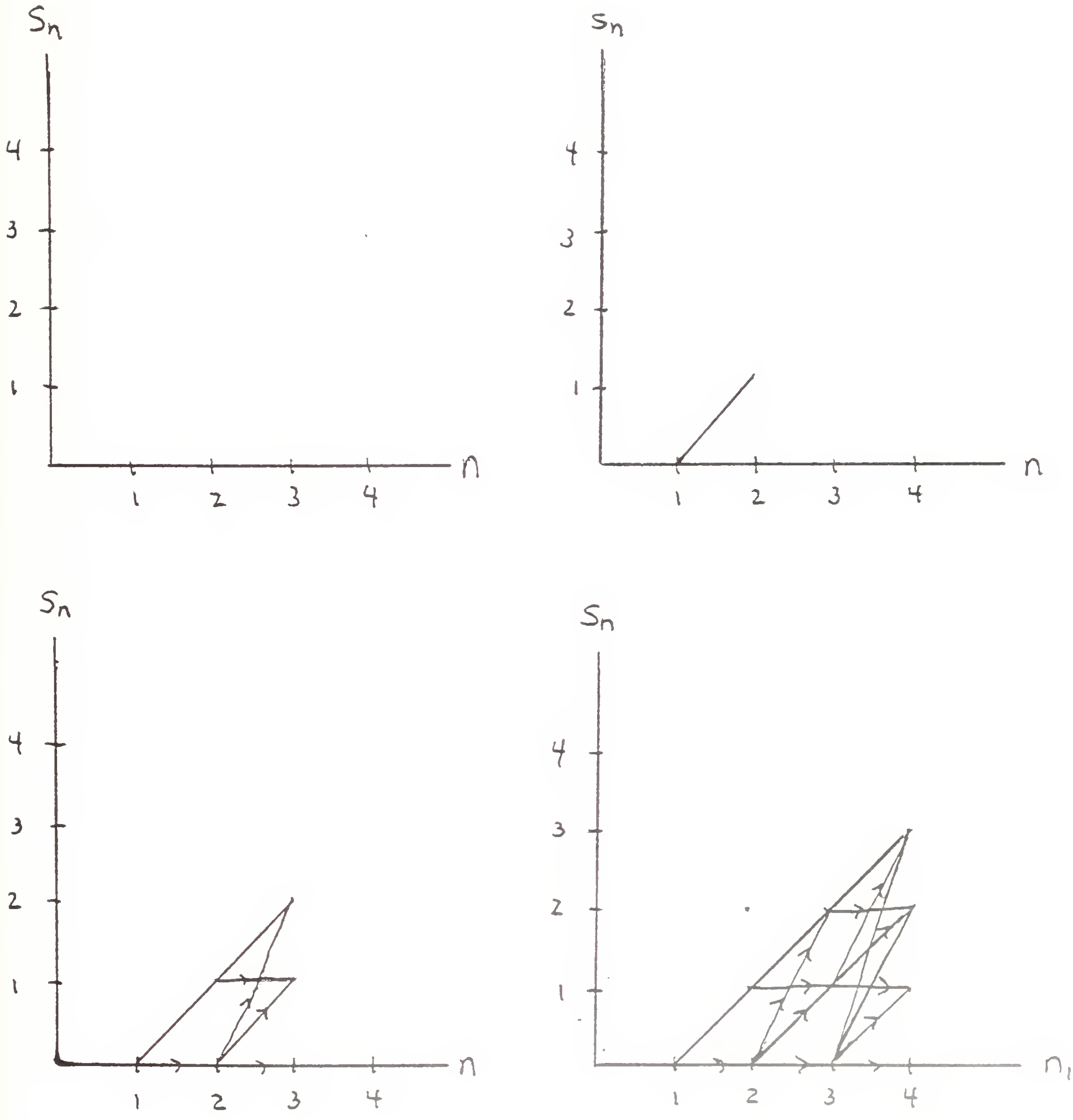
$$P_5 = P(X_1 < 1,\ X_1+X_2 < 2,\ \ldots.\ X_1+X_2+X_3+X_4+X_5 < 5)$$

$$= \sum_{i=1}^{5} c_{5,i-1}\alpha^4(1-\alpha)^{i-1}$$

In general $P_k = \sum_{i=1}^{K} c_{k,i-1}\alpha^k(1-\alpha)^{i-1}$

Now what about the coefficients $c_{k,i}$?

One can analyze the possible paths to a given termination point by an induction method. Consider the cases K=1, 2, 3,4.



For K=1 $P_1 = P(X_1 < 1) = P(X_1 = 0) = \alpha$

For K=2 the two termination points (2,0),(2,1) have only one path to each of them hence $C_{20} = 1$ $C_{21} = 1$ and

$P_2 = \alpha^2 + \alpha^2(1-\alpha)$

For K=3, the three termination points are (3,0) (3,1) (3,2).

Consider (3,0). It can only be reached from (2,0). Since the number of ways from (1,0)→(2,0) = 1, then $C_{30} = C_{20}$

Consider (3,1). It can be reached from (2,0) and 2,1). Hence $C_{31} = C_{20} + C_{21}$. Likewise $C_{32} = C_{20} + C_{21}$.

Thus it appears that one can compute the C_{Ki}'s using a modified form of Pascal's triangle.

<u>K</u>					n=7		
1	1						
2	1	1					
3	1	2	2				
4	1	3	5	5			
5	1	4	9	14	14		
6	1	5	14	28	42	42	
7	1	6	20	48	90	132	132

etc.

where each row starts with 1 and each successive member is equal to the sum of all row members in the preceding row starting from the left up to but not including the same column.

$$\text{i.e.} \quad C_{Ki} = \sum_{j=1}^{i-1} C_{K-1,\, j}$$

Now

$$q_{n-k} = P(S_1 < 0,\ S_2 < 0,\ \ldots .\ S_{n-k} < 0)$$

$$= P(Y_1 < 1,\ Y_2 < 2,\ \ldots .\ Y_{n-k} < n-k)$$

where the $Y_i \sim$ Geometric $\left(P^* = \frac{\mu_1}{\lambda + \mu_1}\right)$.

By analogy

$$q_{n-k} = \sum_{j=1}^{n-k} c_{n-k,\, j-1}\, \beta^{\, n-k} (1-\beta)^{j-1}$$

$$\beta = \left(\frac{\mu_1}{\lambda + \mu_1}\right)$$

Hence $P(S_k \text{ is max}) = p_k q_{n-k}$

$$P(S_k \text{ is max}) = \sum_{i=1}^{k} c_{k,\, i-1}\, \alpha^{k} (1-\alpha)^{i-1} \sum_{j=1}^{n-k} c_{n-k,\, j-1} \cdot$$

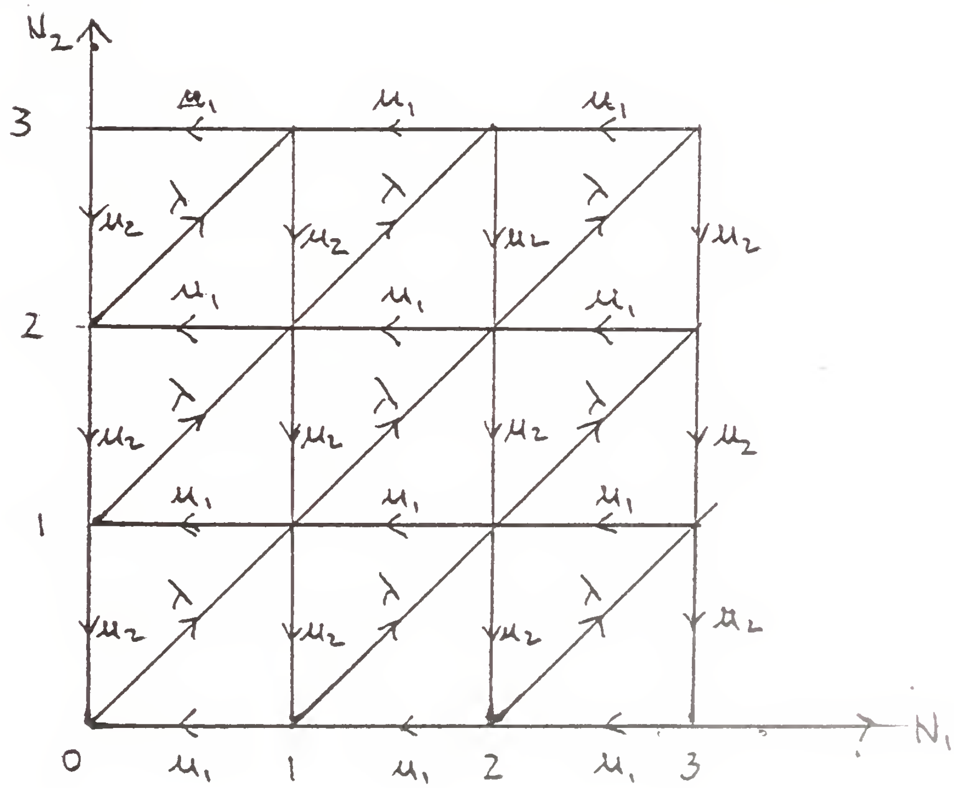
$$\beta^{\, n-k} (1-\beta)^{j-1}$$

To date, no method has been found to simplify the above expression because of the complexity of the coefficients, c_{ij}.

IV. THE TWO CHANNEL SERVER QUEUE WITH SINGLE SERVERS AND LIMITED QUEUE CAPACITY

In this chapter, the system will be further modified by placing a limit on the total number of customers allowed in the system. Thus if N=K, this implies that N_1 and N_2, the number of jobs in each queue, are also limited by K. Since $N = \max \{N_1, N_2\}$ then the ordered pair (N_1, N_2) are restricted to states bounded by a square region in the first quadrant. The diagram below illustrates the allowed transitions for K equal to 3.

N_2



The balance equations for the K capacity system are complicated, but finite in number.

$$(\lambda + \mu_1 + \mu_2) P_{n_1,n_2} = \lambda P_{n_1-1,n_2-1} + \mu_1 P_{n_1+1,n_2} + \mu_2 P_{n_1,n_2+1}$$

$$1 \le n_1 < K \qquad 1 \le n_2 < K$$

$$(\lambda + \mu_1) P_{n_1,0} = \mu_1 P_{n_1+1,0} + \mu_2 P_{n_1,1}$$

$$1 \le n_1 < K$$

$$(\lambda + \mu_2) P_{0,n_2} = \mu_2 P_{0,n_2+1} + \mu_1 P_{1,n_2}$$

$$1 \le n_2 < K$$

$$(\mu_1 + \mu_2) P_{K,n_2} = \lambda P_{K-1,n_2-1} + \mu_2 P_{K,n_2+1}$$

$$1 \le n_2 < K$$

$$(\mu_1 + \mu_2) P_{n_1,K} = \lambda P_{n_1-1,K-1} + \mu_1 P_{n_1+1,K}$$

$$1 \le n_1 < K$$

$$\lambda P_{00} = \mu_1 P_{10} + \mu_2 P_{01}$$

$$\mu_1 P_{K,0} = \mu_2 P_{K,1}$$

$$\mu_2 P_{0,K} = \mu_1 P_{1,K}$$

$$(\mu_1 + \mu_2) P_{K,K} = \lambda P_{K-1,K-1}$$

There has been a great deal of research performed in the area of finding solutions to large, finite state, queueing systems. The purpose of this work was to avoid the use of computer simulation or standard iteration procedures. Instead, an investigation into the special structure of this finite system was conducted. Although a solution was not obtained, several interesting facts were documented for use in further research.

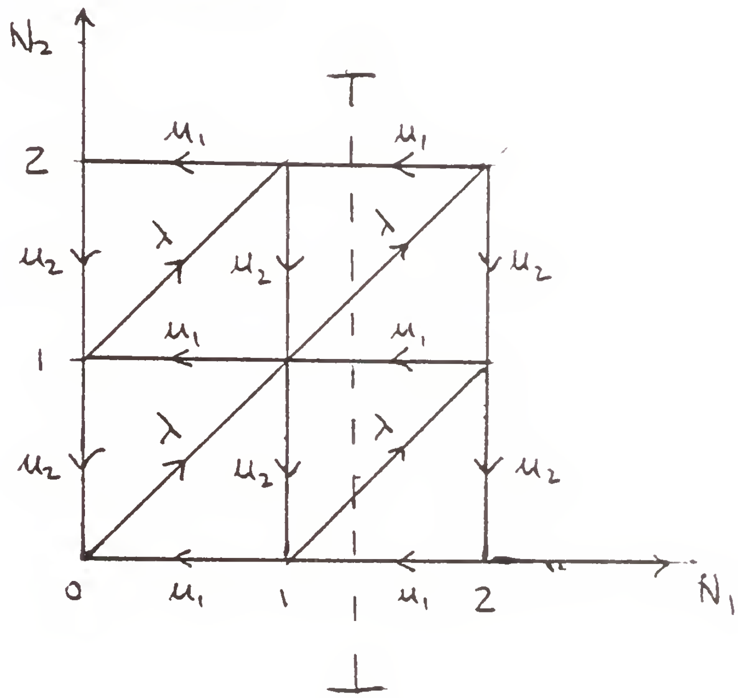
It should be noted that the K capacity system is more difficult to attack because the marginal probability functions for N_1 and N_2 are not known. In the previous chapters, arrival dependency did not affect the marginal distributions. Here, one sees that when either service channel has K jobs, arrivals are turned away.

A. ANALYSIS OF THE TRANSITION DIAGRAM FROM THE VIEWPOINT OF NETWORK FLOW

One method of observing patterns among the steady state probabilities is to view the transition diagram as a network. By slicing the network with a cut, one can equate probability flow across the cut. This is actually doing no more than adding together several balance equations. However, the method does facilitate a rapid development of relationships between

the probabilities. A transition diagram for K equal to two is shown below.

N_2



By equating flow across the cut one finds that

$$\lambda P_{11} + \lambda P_{10} = \mu_1 P_{22} + \mu_1 P_{21} + \mu_1 P_{20}$$

This equation could have been found by adding together the three balance equations for P_{20}, P_{21}, P_{22} as follows.

$$\mu_1 P_{20} = \mu_2 P_{21}$$

$$(\mu_1 + \mu_2) P_{21} = \lambda P_{10} + \mu_2 P_{22}$$

$$+ \quad (\mu_1 + \mu_2) P_{22} = \lambda P_{11}$$

$$\overline{\mu_1 P_{20} + \mu_1 P_{21} + \mu_1 P_{22} = \lambda P_{11} + \lambda P_{10}}$$

Now let us consider an arbitrary value K. For any finite K, the same type of cuts can be performed. If one writes all the equations resulting from vertical cuts such as the one above, it follows that,

$$\lambda \sum_{n_2=0}^{K-1} P_{0,n_2} = \mu_1 \sum_{n_2=0}^{K} P_{1,n_2}$$

$$\lambda \sum_{n_2=0}^{K-1} P_{1,n_2} = \mu_1 \sum_{n_2=0}^{K} P_{2,n_2}$$

$$\vdots$$

$$\lambda \sum_{n_2=0}^{K-1} P_{K-1,n_2} = \mu_1 \sum_{n_2=0}^{K} P_{K,n_2}$$

Using the fact that $P(N_1 = i) = \sum_{n_2=0}^{K} P_{i,n_2}$ for any $0 \leq i \leq K$, one sees that the vertical cuts actually show relationships between the marginal probabilities. In particular,

$$\lambda P(N_1 = 0) - \lambda P_{0,K} = \mu_1 P(N_1=1)$$

$$\lambda P(N_1 = 1) - \lambda P_{1,K} = \mu_1 P(N_1=2)$$

$$\lambda P(N_1=K-1) - \lambda P_{K-1,K} = \mu_1 P(N_1=K)$$

In order to eliminate the P_{ij} terms, one can add all of the above equations to obtain

$$\lambda \left[1 - P(N_1 = K)\right] - \lambda\left[P(N_2 = K) - P_{K,K}\right] = \mu_1 \left[1-P(N_1 = 0)\right]$$

Note that the only probability which is not a marginal is $P_{K,K}$.

In an analagous fashion, all the horizontal cuts in a K-level transition diagram yield the following equation.

$$\lambda\left[1-P(N_2 = K)\right] - \lambda\left[P(N_1 = K) - P_{K,K}\right] = \mu_1\left[1-P(N_2 = 0)\right]$$

Eliminating $P_{K,K}$ from both equations yields the surprising result that

$$\mu_1\left[1-P(N_1 = 0)\right] = \mu_2\left[1 - P(N_2 = 0)\right]$$

At the least, the above relationship could be used as a check against solutions obtained from iteration or simulation. The above results also show that the marginal probabilities may be more efficiently solved for if one knows the boundary probabilities in the original transition network.

Another interesting set of relationships can be found by combining the first two equations found from the vertical cuts. Recall that

$$\lambda P(N_1=0) - \lambda P_{0,K} = \mu_1 P(N_1=1)$$

$$\lambda P(N_1=1) - \lambda P_{1,K} = \mu_1 P(N_1=2)$$

Additionally using the balance equation for node $P_{0,K}$, it follows that $\mu_2 P_{0,K} = \mu_1 P_{1,K}$. Thus the two equations can be combined to yield

$$\frac{\lambda\mu_2}{\mu_1} P(N_1=0) = (\lambda+\mu_2) P(N_1=1) - \mu_1 P(N_1=2)$$

It follows by symmetry that

$$\frac{\lambda\mu_1}{\mu_2} P(N_2=0) = (\lambda+\mu_1) P(N_2=1) - \mu_2 P(N_2=2)$$

Thus one has obtained two equations strictly in terms of the marginal probabilites.

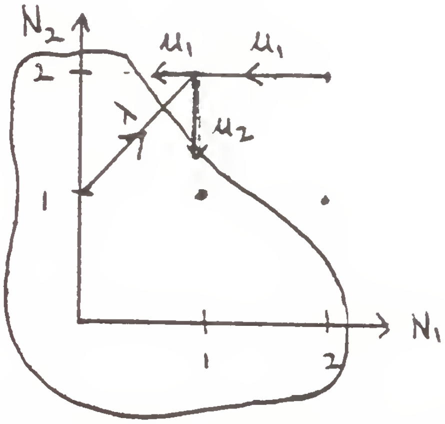
If one continued this pattern, it would not be possible to arrive at an entire system of pure marginal equations. However, the above equations can be used as check against any analytical procedure which might seek solutions to the marginal probabilities. This check is used later in this paper to show that a proposed semi-Markov model for marginal probabilities is incomplete.

Several other cuts were attempted, but the results always appeared to re-establish the previously stated equations.

B. REDUCTION OF THE NUMBER OF BALANCE EQUATIONS BY SUBSTITUTION

Consider again the system for K equal to two. It will be shown that the nine homogeneous balance equations can be reduced to three by a method of substitution. In terms of an arbitrary K, this translates to reduction of $(K+1)^2$

balance equations into a system of 2K-3 balance equations in 2K-3 unknowns. The method is as follows.



In theory, it is possible to express each probability in the set circled in terms of boundary probabilities $P_{2,1}$, $P_{2,2}$ and $P_{1,2}$. In a sense, one is "walking" from one boundary to another on the opposite side of the network. In finding all P_{ij}'s in terms of $P_{2,1}$, $P_{2,2}$ and $P_{1,2}$, the only balance equations which are not utilized are those for $P_{0,1}$, $P_{0,0}$, and $P_{1,0}$. As an example, $P_{0,1}$ can be expressed in terms of $P_{1,2}$ and $P_{2,2}$ by using the balance equation for $P_{1,2}$, that is

$$(\mu_1+\mu_2)\ P_{1,2} = \lambda P_{0,1} + \mu_1 P_{2,2}$$

$$P_{0,1} = \frac{\mu_1+\mu_2}{\lambda}\ P_{1,2} - \frac{\mu_1}{\lambda}\ P_{2,2}$$

Thus if all the P_{ij}'s are expressed in terms of P_{K,n_2} where $1 \leq n_2 \leq K$, $P_{n_1,K}$ where $1 \leq n_1 < K$, and $P_{K,K}$, one can use the remaining boundary equations for $P_{n_1,0}$ $1 \leq n_1 < K$, P_{0,n_2} $1 \leq n_2 < K$ and $P_{0,0}$ to arrive at a homogeneous system of three equations in three unknowns. For the above case,

$$P_{0,2} = \frac{\mu_1}{\mu_2} P_{1,2}$$

$$P_{0,1} = \frac{\mu_1 + \mu_2}{\lambda} P_{1,2} - \frac{\mu_1}{\lambda} P_{2,2}$$

$$P_{0,0} = \frac{(\lambda + \mu_1 + \mu_2)(\mu_1 + \mu_2)}{\lambda^2} P_{2,2} - \frac{\mu_1}{\lambda} P_{2,1} - \frac{\mu_2}{\lambda} P_{1,2}$$

$$P_{1,0} = \frac{\mu_1 + \mu_2}{\lambda} P_{2,1} - \frac{\mu_2}{\lambda} P_{2,2}$$

$$P_{2,0} = \frac{\mu_2}{\mu_1} P_{2,1}$$

and $P_{1,1} = \dfrac{\mu_1 + \mu_2}{\lambda} P_{2,2}$

Now if one writes the balance equations for nodes $P_{0,1}$, $P_{0,0}$, and $P_{1,0}$ using the above equations, the result will be three homogeneous equations in three unknowns, $P_{2,1}$, $P_{2,2}$, and $P_{1,2}$.

It must be pointed out that there did not appear to be any obvious pattern to the build up of coefficients as one "walks" through the network. All attempts made to find closed form expressions for the coefficients were to no avail. In theory, each P_{ij} can be expressed as

$$P_{i,j} = \sum_{m=1}^{K-1} C_m P_{K,m} + \sum_{n=1}^{K-1} d_n P_{n,K} + \alpha P_{K,K}$$

The above substitution is no different from a Gaussian elimination scheme that could be used for any system of equations. However, it was hoped that these equations contained special structure and would lend themselves to an analytical solution.

C. COMMENTS CONCERNING THE MATRIX STRUCTURE OF THE BALANCE EQUATIONS

One can write all the balance equations in the form $\underline{A}\underline{X}=\underline{B}$ where $\underline{A}$ is $(K+1)^2$ by $(K+1)^2$, $\underline{X}$ is the $(K+1)^2$ by 1 column vector for the P_{ij}'s and $\underline{B}$ is a $(K+1)^2$ by 1 column vector which has all but one element equal to 0. The result is a system of $(K+1)^2-1$ homogeneous equations and 1 normality equation $\sum_{ij} P_{ij} = 1$. The normality equation may replace any one of the original $(K+1)^2$ homogeneous equations and thus insure that the non-homogenous system has a unique solution.

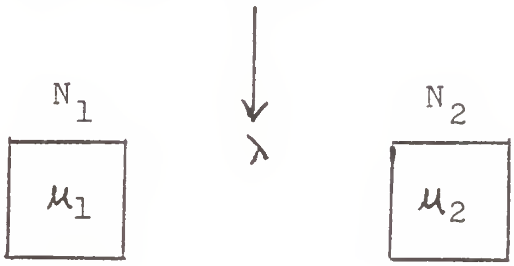
After a great deal of experimentation with reordering the rows and columns of the coefficient matrix, it was concluded that there existed no special structure in the equation set. At first glance, one might be attracted to the regularity of the coefficient matrix. As an example writing the balance equations for K=2 in descending order of the second and then first coordinate one arrives at the following system.

$$
\begin{bmatrix}
\mu_1+\mu_2 & & & & -\lambda & & & & \\
-\mu_2 & \mu_1+\mu_2 & & & & -\lambda & & & \\
 & -\mu_2 & \mu_1 & & & & & & \\
-\mu_1 & & & \mu_1+\mu_2 & & & & -\lambda & \\
 & -\mu_1 & & -\mu_2 & \lambda+\mu_1+\mu_2 & & & & -\lambda \\
 & & -\mu_1 & & -\mu_2 & \lambda+\mu_1 & & & \\
 & & & -\mu_1 & & & \mu_2 & & \\
 & & & & -\mu_1 & & -\mu_2 & \lambda+\mu_2 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} P_{22} \\ P_{21} \\ P_{20} \\ P_{12} \\ P_{11} \\ P_{10} \\ P_{02} \\ P_{01} \\ P_{00} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

The coefficient matrix is strongly diagonal in structure. However, no special method was found to solve the above system without resorting to the standard elimination schemes already known. In all cases, it was necessary to completely triangularize the matrix and then use back substitution to solve for the P_{ij}'s.

Originally the intent was to find a partial inversion or partial row elimination scheme which would allow for the solution of the boundary probabilities without having to use all the information in the matrix. The conclusion is that the boundary conditions form an inseparable portion of the entire system. Perhaps further research may uncover an efficient scheme for solution.

D. AN ATTEMPT TO SOLVE FOR THE MARGINAL PROBABILITIES USING A SEMI-MARKOV MODEL



Let N_1 = # of customers in queue #1.

Consider just the N_1 queue (including customer in service).

N_1 is dependent upon the arrival stream, the service distribution, and the distribution of N_2. Specifically, when $N_2 = N$, arrivals cannot enter either queue.

The N_1 queue also has capacity = N.

Consider modeling the N_1 queue as a semi-Markov process. The state of the system will be the number of customers in the system. $N_1=0,1,2,\ldots.N$.

Let P_{ij} be the transition probability from $i \to j$.

Let αi = expected time system is in state i.

Let πi = limiting transition probability of transitioning into state i.

Let P_i = steady state probability of being in state i.

Transitions will only be allowed from

$i \to i+1$ and $i \to i-1$.

Using a semi-Markov model, one assumes that when the system is in state i it will remain in state i for a random amount of time (mean = αi). Then the system will either transition

to state i+1 or i-1. When i=0 it will transition to i=1 with probability 1. When i=N transition i=N-1 with probability 1.

In essence, the semi-Markov model has transitions which are in accordance with a Random Walk Markov Chain (with pure reflecting barriers).

Thus the transitions are described as

	0	1	2	3	4	...	N-1	N
0	0	1	0	0	0	...	0	0
1	P_1	0	q_1	0	0	...	0	0
2	0	P_2	0	q_2	0	...	0	0
3	0	0	P_3	0	q_3	...	0	0
4							0	
⋮		0						
N-1						P_{n-1}	0	q_{n-1}
N	0	0	0	0	0	...	1	0

where $p_i + q_i = 1$

The underlying theory of this model allows the following statements.

If π_i i=0,1,2... N are the solutions to the following system.

$$\sum_{i=0}^{N} \pi_i = 1$$

$$\pi_i = \sum_{j=0}^{N} P_{ji} \pi_j$$

Then the P_i's can be found by

$$P_i = \frac{\pi_i \alpha_i}{\sum_{j=0}^{N} \pi_j \alpha_j}$$

The basic requirement for a semi-Markov model is that the distribution of time spent in a state be independent from state to state. It appears as though the exponential character of the service and arrival distributions will insure this.

It should be noted that all π_i's and P_i's must be conditioned on the state of the N_2 queue. A Bernoulli type conditioning will be used. The states of N_2 will be partitioned into two classes $N_2 < N$ $\quad P(N_2 < N) = 1\text{-}K$

$$N_2 = N \qquad P(N_2 = N) = K$$

1. Transition Probabilities

Consider $i = 1, 2, 3, \ldots N-1$

Case I $N_2 < N$

Given the system is in state i

$$P_{i,i+1} = \frac{\lambda}{\lambda + \mu_1} \quad \text{arrival occurs before service}$$

$$P_{i,i-1} = \frac{\mu_1}{\lambda + \mu_1} \quad \text{service occurs before arrival.}$$

Case II $N_2 = N$

When N_2 is capacitated, the transition from i to i+1 can only take place if N_2 serves one customer and then an arrival occurs before N_1 can serve one customer.

i.e. $S_1 \sim EXP(\mu_1)$ $S_2 \sim EXP(\mu_2)$ $A \sim EXP(\lambda)$

$$P_{i,i+1} = P(S_1 \geq S_2+A) = P(S_2 < S_1) \cdot P(A < S_1^*)$$

where S_1^* represents memoryless service

$$P_{i,i+1} = \frac{\mu_2}{\mu_1+\mu_2} \cdot \frac{\lambda}{\lambda+\mu_1} = \frac{\lambda\mu_2}{(\lambda+\mu_1)(\mu_1+\mu_2)}$$

Likewise transition $i \rightarrow i-1$ takes place only when $S_1 \leq S_2+A$

$$P_{i,i-1} = P(S \leq S_2+A) = P(S_1 < S_2) + P(S_1 > S_2,\ S_1^* < A)$$

$$P_{i,i-1} = \frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_2}{\mu_1+\mu_2} \frac{\mu_1}{\lambda+\mu_1} = \frac{\mu_1(\lambda+\mu_1+\mu_2)}{(\lambda+\mu_1)(\mu_1+\mu_2)}$$

Thus finding the total transition probabilities as a function of K, one has

$$P_{i,i+1} = \frac{\lambda}{\lambda+\mu_1}(1-K) + \frac{\lambda\mu_2}{(\lambda+\mu_1)(\mu_1+\mu_2)}K$$

$$P_{i,i-1} = \frac{\mu_1}{\lambda+\mu_1}(1-K) + \frac{\mu_1(\lambda+\mu_1+\mu_2)}{(\lambda+\mu_1)(\mu_1+\mu_2)}K$$

Simplifying

$$P_{i,i-1} = \frac{\mu_1(\lambda K+\mu_1+\mu_2)}{(\lambda+\mu_1)(\mu_1+\mu_2)} = P$$

$$P_{i,i+1} = \frac{\lambda(\mu_1+\mu_2-\mu_1 K)}{(\lambda+\mu_1)(\mu_1+\mu_2)} = q$$

For $i = 0$ $P_{0,1} = 1$ $i = N$ $P_{N,N-1} = 1$

2. Limiting Transition Probabilities

Using the relationship $\pi_i = \sum_{j=0}^{N} P_{ji} \pi_j$

the following system of equations develop.

$$\pi_N = q\pi_{N-1}$$

$$\pi_{N-1} = \pi_N + q\pi_{N-2}$$

$$\pi_{N-2} = q\pi_{N-3} + P\pi_{N-1}$$

$$\pi_{N-3} = q\pi_{N-4} + P\pi_{N-2}$$

$$\vdots$$

$$\pi_3 = q\pi_2 + P\pi_4$$

$$\pi_2 = q\pi_1 + P\pi_3$$

$$\pi_1 = \pi_0 + P\pi_2$$

$$\pi_0 = P\pi_1$$

Setting the equations as in a matrix, one row operation through the matrix produces the following solutions

$$\pi_i = \frac{1}{P}\left(\frac{q}{P}\right)^{i-1}\pi_0 \qquad i=1,2,3, \ldots N-1$$

$$\pi_N = \left(\frac{q}{P}\right)^{N-1}\pi_0$$

Using the normality relationship $\sum_{i=0}^{N} \pi_i = 1$

$$\pi_0 = \frac{2P-1}{2\left(P-q\left(\frac{q}{p}\right)^{N-1}\right)} \qquad P \neq q$$

3. <u>Expected Time in Each State</u>.

The α_i's are calculated by conditioning on N_2

Case I $N_2 < N$

$i = 0$ $\alpha_i = \frac{1}{\lambda}$

$i = 1, 2, \ldots N-1$ $\alpha_i = \frac{1}{\lambda + \mu_1}$

$i = N$ $\alpha_i = \frac{1}{\mu_1}$

Case II $N_2 = N$

$i = 0$ the N_1 queue must wait for 1 N_2 service and then 1 arrival

$$\alpha_i = \frac{1}{\lambda} + \frac{1}{\mu_2}$$

$i = 1, 2, \ldots . N-1$ the N_1 queue waits for the minimum of 1 N_1 service or 1 N_2 service and 1 arrival.

Let $X = \min \left\{ S_1, S_2 + A \right\}$

By conditioning etc.

$$E(X) = \frac{\lambda(\lambda + \mu_1) - \mu_2(\mu_1 + \mu_2)}{(\lambda - \mu_2)(\lambda + \mu_1)(\mu_1 + \mu_2)}$$

i=N

$\alpha_i = \frac{1}{\mu_1}$ still.

Hence the total expected waiting times are

$$\alpha_i = \begin{cases} \dfrac{\lambda K + \mu_2}{\mu_2} & i = 0 \\[2ex] \dfrac{\lambda K + \mu_1 + \mu_2}{(\lambda + \mu_1)(\mu_1 + \mu_2)} & i = 1, 2, \ldots N-1 \\[2ex] \dfrac{1}{\mu_1} & i = N \end{cases}$$

Knowing π_i's and α_i's one can calculate closed form expressions for P_i's in terms of λ, μ_1, μ_2, and K.

Now from original two dimensional steady state transition diagram one can show the following relationship to be true. This equation was derived in the previous section.

$$(\lambda + \mu_2)\, P_1 = \mu_1\, P_2 + \frac{\lambda \mu_2}{\mu_1}\, P_0$$

For the case, K=3, the above equation could not be satisfied using the expressions for P_0, P_1, P_2 found by the proposed model. It appears as though the fixed coupling coefficient K does not completely characterize the interaction between the two queues. At least having the above marginal equation available allowed for a quick check against the model.

V. CONCLUSION

There is a great deal more research which can be performed in order to better understand the effects of arrival dependence upon service systems.

As one can no doubt conclude, finding analytical solutions to queueing problems of this type is extremely difficult and may in fact require a completely novel approach.

The infinite server queue did not pose much difficulty because the waiting time of any customer was strictly a function of his own service time. However, the systems with finite servers and finite capacity still remain unsolvable from an analytical viewpoint.

One can think of many instances where finite server and finite capacity queues could be used to model repair and maintenance systems. Consider a military maintenance company which services vehicles or generators. Often, the arrival of a certain item of equipment initiates several activities, many of which are independent. It was the aim of this research to model such activities. Unfortunately, only partial answers have been obtained. Hopefully, more study

will be concentrated in this area of stochastic modeling in order to advance the current knowledge of these and other related systems.

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